Design of Algebraic Observers for Brass Instruments

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July 11, 2014

Introduction

This paper addresses the problem of elaborating a state observer for a simplified brass instrument model, assuming that the output is the measured pressure at the end of the pipe. The observer is based on delay-differential-algebraic-equations (DDAE) and tools from Kalman filter theory in the case of correlated noises (see (Ma, Wang, and Chen 2010)).

Section 2 introduces a simplified model for the instrument. The existence and computation of the solutions of the model in DDAE form is presented in section 3. Section 4 is dedicated to the design of our observer. First simulation results are presented which validate our approach. Finally in section 5 the sensitivity analysis from the measured output to the position of the lip is detailed.

Modelling

A simplified brass instrument can be described by:

- a valve, the aperture of which is modulated by a single solid, namely a lip, characterized by its mass, stiffness and damping. The bottom of the lip is moving in the vertical direction and located by its height $\xi(t)$,
- a quasi-steady jet which applies a force on the valve,
- an acoustic pipe, the vibrations of which are set in motion by the jet.

The reader can refer to (d'Andréa-Novel, Coron, and Hélie 2010) for more details.

Acoustic Tube and Jet Dynamics

Let $p_m(t)$ be the pressure in the musician's mouth at time t (all pressures considered here are relative to the atmospheric pressure). We use the convention that values that depend on a time variable t and a space variable x are implicitly evaluated at the origin ($x = 0$) when the space variable is not specified.

Let $p(t, x)$ be the acoustic pressure in the resonator at time t and point x. This pressure is a superposition of forward and backward pressure waves $p^+(t, x)$ and $p^-(t, x)$: at any time and location in the tube, we have $p(t, x) = p^+(t, x)+p^-(t, x)$ and in particular, at the origin

$$p(t) = p^+(t) + p^-(t). \tag{1}$$

When the lip is open – that is $\xi(t) > 0$ – a simple modelling based on the Bernoulli equation leads to the following relation between pressures

$$\frac{\mu}{2}\left(\frac{p^+(t) - p^-(t)}{\xi(t)}\right)^2 = p_m(t) - p^+(t) - p^-(t). \tag{2}$$

We refer to (d'Andréa-Novel, Coron, and Hélie 2010) for the derivation of this equation from the modelling assumptions and the expression of the constant μ in terms of the physical parameters of the system.

At the other end of the tube, we assume that pressure waves are reflected by a constant, frequency-independent, parameter $\lambda \in (-1, +1)$. Consequently, if τ denotes the time for the sound to travel from the tube origin to the extremity and back, ℓ the length of the tube and c the celerity of the waves, we have

$$p^-(t) = \lambda p^+(t - \tau) \;\text{ with }\; \tau = \frac{2\ell}{c}. \tag{3}$$

Given that $p^+(t, x)$ and $p^-(t, x)$ are progressive waves,

$$p^+(t, x) = p^+(t - x/c) \;\text{ and }\; p^-(t, x) = p^-(t + x/c). \tag{4}$$

The measured output of the system is the sound radiated at the end of the instrument, that is:

$$y(t) = p(t, \ell) = p^+(t - \ell/c) + p^-(t + \ell/c) \tag{5}$$

or, using relation (3)

$$y(t) = (1 + \lambda)p^+(t - \tau/2). \tag{6}$$

Lip Dynamics

Lips are modelled as solid masses subject to pressure, friction and elastic forces. Spring-damper systems are classically described by linear second-order differential equations under the canonical representation

$$\ddot{\xi} + 2\zeta\omega\dot{\xi} + \omega^2(\xi - \xi_e) = \omega^2 f \tag{7}$$

where ω is the natural frequency, ζ the damping ratio and ξ_e the equilibrium position. The system input f is a linear combination of side pressure (difference) and jet pressure:

$$f = \gamma_s(p_m - p) + \gamma_j p. \tag{8}$$

The natural frequency ω and the damping ratio ζ in (7) are related to the upper lip mass m, damping a and stiffness k by $\omega = \sqrt{k/m}$ and $\zeta = a/(2\sqrt{km})$. The side pressure gain and jet pressure gain γ_s and γ_j are deduced from geometric parameters of the upper lip – the angle of incidence θ and side and bottom area A_s and A_b – with $\gamma_s = (A_s \sin\theta)/k$ and $\gamma_j = A_b/k$.

Remark. In the scope of this article, only a single-lip model, with a lip that is always open is considered. The open-closed model behavior is studied in (d'Andréa-Novel, Coron, and Hélie 2010) and more refined models, such as two-dimensional lip models (see e.g. (Adachi and Sato 1996)) used for trumpet sound synthesis could also be considered.

Structure of the Dynamics

Input Pressure Validity

We obviously expect the instrumentist to apply a mouth pressure that produces a sustained sound. However, note that some values of this input pressure may generate no admissible solution in our model. For example, equation (2) clearly requires

$$p_m(t) \geq p^+(t) + p^-(t), \tag{9}$$

in other words, that at any time t, the pressure $p(t)$ at the origin of the acoustic tube shall be smaller than the mouth pressure. Otherwise, no solution can exist.

This condition however is not sufficient to ensure the existence of solutions; it does not enforce either the satisfaction of another sensible assumption[1]: that

[1] this assumption is derived in (d'Andréa-Novel, Coron, and Hélie 2010) under the hypotheses of airflow conservation and of a lip aperture size much smaller than the mouth section.

the air always flows from the mouth to the pipe, a condition equivalent to $p^+(t) - p^-(t) \geq 0$.

To analyze this issue, we rewrite equation (2) as a quadratic equation $a\Delta p^2 + b\Delta p + c = 0$ whose unknown is $\Delta p = p^+(t) - p^-(t)$, and with coefficients $a = \mu/2\xi(t)^2$, $b = 1$, $c = 2p^-(t) - p_m(t)$. As $a > 0$ and $b > 0$, the sum of the two solutions of this equation is negative, hence – if the solutions are real – at least one of them is negative. Hence, at most one solution can be consistent with a nonnegative airflow; this solution does exist if the product of solutions is nonpositive, which holds iff the product ac is nonpositive, that is:

$$p_m(t) \geq 2p^-(t). \tag{10}$$

If this condition holds, there is a single, real, nonnegative solution $p^+(t) - p^-(t)$ to (2) and the condition (9) is automatically satisfied.

Functional Differential Equations

Under the assumption that (10) holds at every instant, we may compute the forward pressure $p^+(t)$ as

$$p^+(t) = P^+(p^+(t-\tau), \xi(t), p_m(t)) \tag{11}$$

while the lips dynamics is structured as

$$\ddot{\xi}(t) = L(p^+(t-\tau), \xi(t), \dot{\xi}(t), p_m(t)). \tag{12}$$

Relations (11) and (12) constitute a system of delay-differential algebraic equations (DDAE, see for example (Boisgérault 2013) for an introduction). Its state at time t is a triple $(p_t^+, \xi(t), \dot{\xi}(t))$ where p_t^+ denotes the function defined for $\theta \in [-\tau, 0]$ by $p_t^+(\theta) = p^+(t+\theta)$.

These systems are frequently reduced to neutral delay-differential equations (NDDE), obtained by the differentiation of the algebraic component of their dynamics, here equation (11). This strategy is applied in (d'Andréa-Novel, Coron, and Hélie 2010) where the neutral form is used to compute approximate solutions of the system and also as a basis for the design of observers. This approach is common[2] but generates some restrictions. The most obvious one is that the differentiation is only valid under some regularity assumptions: the mouth pressure p_m and the initial p_0^+ should be absolutely continous, assumptions that are not always realistic. Even if these conditions are met – for example

[2] the use of a dynamic Bernoulli equation instead of the static one or of frequency-dependent impedance at the acoustic tube boundary for example would require the modelling as a neutral system.

for constant mouth pressure p_m and initial forward pressure wave p^+ – this derivation validity is still subject to splicing conditions (see e.g. (Bellen and Zennaro 2003)) and hence may require the modification of the prescribed initial conditions.

Therefore, to avoid those complexities, we have designed a dedicated solver instead, that computes the solutions to this system in its original DDAE form. We use a simple explicit Euler scheme but with a high-frequency resolution, well above the few kHz that would normally be required to represent the sound output by the model with a high fidelity. This configuration is necessary to manage with sufficient accuracy the very oscillatory behavior of the lip that operates near its natural frequency.

Observer Design

A complete estimation of a brass instrument state was provided in the earlier work (d'Andréa-Novel, Coron, and Hélie 2010) in a similar context; the observer design was based on a representation of the full dynamics as a neutral system and the use of Lyapunov methods. We have already pointed out that we favored here a DDAE model of the system dynamics instead and in this section, we will actually take advantage of this representation. Our approach also focuses on the estimation of the lips state, which is sufficient to recover the instrumentist control parameters. As a consequence, we design an observer whose complexity is greatly reduced. Finally, we explicitely model the precision of the output sound measurement to design an observer that is adapted to this setting instead of an observer that is merely robust with respect to the presence of noise in the measure.

Direct Lip Height Estimation

Functional Dependency

The Bernouilli equation (2) and the expressions of the backward pressure wave (3) and of the output pressure (6) can be combined to explicit the dependency between the lip height and the output pressure. We obtain

$$\xi(t)^2 = \Xi^2(p_m(t), y(t-\tau/2), y(t+\tau/2)) \tag{13}$$

where Ξ^2 is the real-valued function of the time and of the delayed and advanced output pressure defined by:

$$\Xi^2(p_m, y_-, y_+) := \frac{\mu}{2(1+\lambda)} \times \frac{(y_+ - \lambda y_-)^2}{(p_m - y_+) + \lambda(p_m - y_-)} \tag{14}$$

Note that the right-hand side of equation (14) is well defined when the mouth pressure is greater than the pressure at the origin of the pipe, a condition already pointed out as necessary for the existence of solutions in our model.

Sensitivity Analysis

From now on, we assume that the exact value of the output pressure $y(t)$ is not available and that only an approximation of it – denoted $\widehat{y}(t)$ – can be used in our computations. Consequently, the exact lip height cannot be computed, but can be estimated with the formula

$$\widehat{\xi}(t)^2 = \Xi^2(p_m(t), \widehat{y}(t - \tau/2), \widehat{y}(t + \tau/2)) \tag{15}$$

A first-order approximation of the the error between this estimate and the exact value $\xi(t)$ is given by

$$\widehat{\xi}(t) - \xi(t) \simeq \nabla_y \Xi(t) \cdot \left[\begin{array}{c} \widehat{y}(t - \tau/2) - y(t - \tau/2) \\ \widehat{y}(t + \tau/2) - y(t + \tau/2) \end{array} \right] \tag{16}$$

where $\nabla_y \Xi(t)$ is a compact notation that represents the gradient of Ξ with respect to the variables (y_-, y_+), evaluated at the point $(p_m(t), y(t - \tau/2), y(t + \tau/2))$. The computation of $\nabla_y \Xi$ is carried out in section .

Discrete and Stochastic Model

Discrete-Time Lips Dynamics

Let X denote the lips state vector – lip height and velocity – but $\tau/2$ seconds in the past:

$$X(t) = \left[\begin{array}{c} \xi(t - \tau/2) \\ \dot{\xi}(t - \tau/2) \end{array} \right]. \tag{17}$$

The application of the Euler explicit scheme with step size dt to the lips dynamics leads to

$$X(t + dt) = AX(t) + Bp(t - \tau/2) + u(t) \tag{18}$$

where

$$A = \left[\begin{array}{cc} 1 & dt \\ -\omega^2 dt & 1 - 2\zeta\omega dt \end{array} \right], \; B = \left[\begin{array}{c} 0 \\ \omega^2 dt(\gamma_j - \gamma_s) \end{array} \right] \tag{19}$$

and

$$u(t) = \left[\begin{array}{c} 0 \\ \omega^2 dt[\gamma_s p_m(t-\tau/2)+\xi_e] \end{array} \right] \tag{20}$$

Output Noise Model

We model the difference between the measure $\hat{y}(t)$ of the output sound and the exact value $y(t)$ as an additive white gaussian noise $n(t)$ of constant variance Σ_y:

$$\hat{y}(t) = y(t) + n(t).$$

The measure $\hat{y}(t)$ provides some approximate value $\hat{z}(t)$ of $z(t) = \xi(t-\tau/2) = CX(t)$ with $C = [1, 0]$. Indeed, using equation (16), we end up with

$$\hat{z}(t) \simeq CX(t) + v(t) \tag{21}$$

with

$$v(t) = \nabla_y \Xi(t-\tau/2) \cdot \left[\begin{array}{c} n(t-\tau) \\ n(t) \end{array} \right]. \tag{22}$$

Given the properties of $n(t)$, the perturbation $v(t)$ has an autocovariance equal to zero for shifts smaller than τ. For the sake of simplicity, we postulate that this autocovariance is actually always zero for every non-zero time shift, effectively modelling $v(t)$ as a gaussian white noise. Its variance is given by

$$\Sigma_v(t) = \|\nabla_y \Xi(t-\tau/2)\|^2 \times \Sigma_y. \tag{23}$$

It should be noted that despite a constant output sound noise power, the standard deviation of $v(t)$ is time-dependent. Experiments, reproduced in the figure "Lip height estimation and standard deviation", demonstrate that this formula provides a good approximation of the variance of the height measure, even when the value of $\nabla_y \Xi$ is an approximation based on noisy data (see section).

State Disturbance

The combination of equations (1), (3) and (6) provides

$$p(t-\tau/2) = \frac{y(t) + \lambda y(t-\tau)}{1+\lambda}. \tag{24}$$

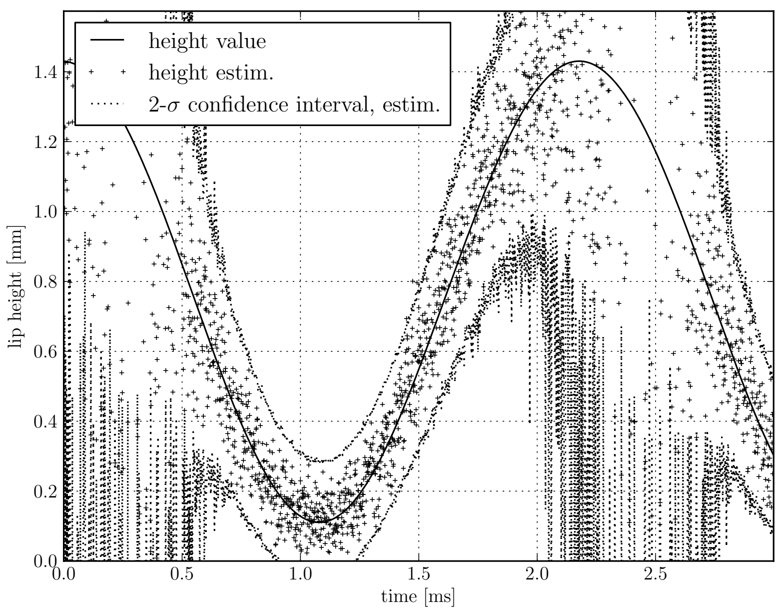

Figure 1: Lip height estimation and standard deviation

The substitution of y by $\hat{y}$ in this equation leads to the definition of a measurable approximation $\hat{p}(t-\tau/2)$ of $p(t-\tau/2)$. The right-hand side of (18) can then be expressed as

$$X(t+dt) = AX(t) + B\hat{p}(t-\tau/2) + u(t) + w(t) \tag{25}$$

where $w(t) = [w_1(t), w_2(t)]^t$ with $w_1(t) = 0$ and

$$w_2(t) = -\omega^2 dt(\gamma_j - \gamma_s)\frac{n(t) + \lambda n(t-\tau)}{1+\lambda}. \tag{26}$$

Using an approximation similar to the one already made for $v(t)$, we model $w(t)$ as a gaussian white noise with covariance matrix

$$\Sigma_w = \omega^4 dt^2 (\gamma_j - \gamma_s)^2 \frac{1+\lambda^2}{(1+\lambda)^2} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_y \end{bmatrix}. \tag{27}$$

Note that the defining equations (22) and (26) also yield a zero covariance between $v(t)$ and $w(t+\theta)$ for any $0 < |\theta| < \tau$. Again, we will assume that we can extend that assumption to any $|\theta| > 0$. However we do neglect the covariance between $v(t)$ and $w(t)$, that is significant: we have $\Sigma_{vw}(t) = [0, \Sigma_{vw_2}(t)]$ with

$$\Sigma_{vw_2}(t) = -\omega^2 dt(\gamma_j - \gamma_s)\Sigma_y \times \nabla_y \Xi(t-\tau/2) \cdot \begin{bmatrix} \lambda/(1+\lambda) \\ 1/(1+\lambda) \end{bmatrix}. \tag{28}$$

Kalman Filter

Principles Equations (21) and (25), together with the stochastic modelling of $v(t)$ and $w(t)$, describe a system whose state can be optimally estimated by a Kalman filter. The measure and state noise of the dynamics are correlated, a particularity that can be managed by the methods exposed in [**?**].

The Kalman filter tracks the evolution of two values: the state estimate at time t denoted $X(t|t)$ and the corresponding estimation error $P(t|t) = \mathbf{cov}[X(t) - X(t|t)]$. The state estimate $X(t|t)$ is defined as the conditional expectation of $X(t)$ with respect to $y(t), y(t-dt), ..., y(0)$.

These values are *a posteriori* values that integrate the information given by the measure $\hat{z}(t)$. The *a priori* estimates of the state and error that use only the information available up to the time $t-dt$ are denoted $X(t|t-dt)$ and $P(t|t-dt)$. The two sets of values at time t are related by the projection formula

$$X(t|t) = X(t|t-dt) + K(t)(y(t) - CX(t|t-dt)) \tag{29}$$
$$P(t|t) = P(t|t-dt)(I - K(t)CP(t|t-dt)) \tag{30}$$

where

$$K(t) = P(t|t-dt)C^t[CP(t|t-dt)C^t + \Sigma_v(t)]^{-1}. \tag{31}$$

As in [?], we introduce $w'(t) = w(t) - J(t)v(t)$ with

$$J(t) = \Sigma_{vw}(t)\Sigma_v(t)^{-1}. \tag{32}$$

This auxiliary random vector is not correlated with $v(t)$. The computation of the *a priori* values at time t yields:

$$\begin{aligned} X(t+dt|t) = {} & AX(t|t) + B\hat{p}(t-\tau/2) + u(t) \\ & + J(t)[\hat{z}(t) - CX(t|t)] \end{aligned} \tag{33}$$

$$P(t+dt|t) = [A - J(t)C]P(t|t)[A - J(t)C]^t + \Sigma_{w'}(t) \tag{34}$$

where

$$\Sigma_{w'}(t) = \Sigma_w(t) - \Sigma_{wv}(t)\Sigma_v(t)^{-1}\Sigma_{vw}(t). \tag{35}$$

Concrete Design The implementation of this estimator shall address the fact that the covariances $\Sigma_v(t)$ and $\Sigma_{wv}(t)$ used in the computations are not available. Indeed, both values depend on the gradient $\nabla_y \Xi(t-\tau/2) = \nabla_y \Xi(p_m(t-\tau/2), y(t-\tau), y(t))$ and the exact values of y are not available, only the noisy measurement $\hat{y}$. Our strategy is – in all occurrences of $\nabla_y \Xi$ in the filter computations – to replace y with $\hat{y}$.

However, this substitution has a drawback. We know that the exact acoustics pipe model produces an output sound $y(t)$ such that at every instant t, the inequalities $y(t) > \lambda y(t-\tau)$ and $(1+\lambda)p_m(t-\tau/2) > y(t) + y(t-\tau)$ hold, hence $\nabla_y \Xi(t)$ is always properly defined. But there is no such guarantee with the noisy data: some values $(p_m, \hat{y}_-, \hat{y}_+)$ may be out of the domain of definition of $\nabla_y \Xi$. In such circumstance, we adopt the following method: we set $\Sigma_v(t) = +\infty$ and $\Sigma_{vw}(t) = [0, 0]$. This choice effectively discards the measure of the output sound from the Kalman filter equations. The *a posteriori* values $X(t|t)$ and $P(t|t)$ are set equal to the *a priori* values $X(t|t-dt)$ and $P(t|t-dt)$. On the other hand the update of the state and state error estimate are given by $X(t+dt|t) = AX(t|t) + B\hat{p}(t-\tau/2) + u(t)$ and $P(t+dt|t) = AP(t|t)A^t + \Sigma_w(t)$.

Simulation Results Experiments demonstrate that despite the set of approximations used in its design, the Kalman filter performs an efficient estimation of the lips state even in the presence of large output noise. The steady-state

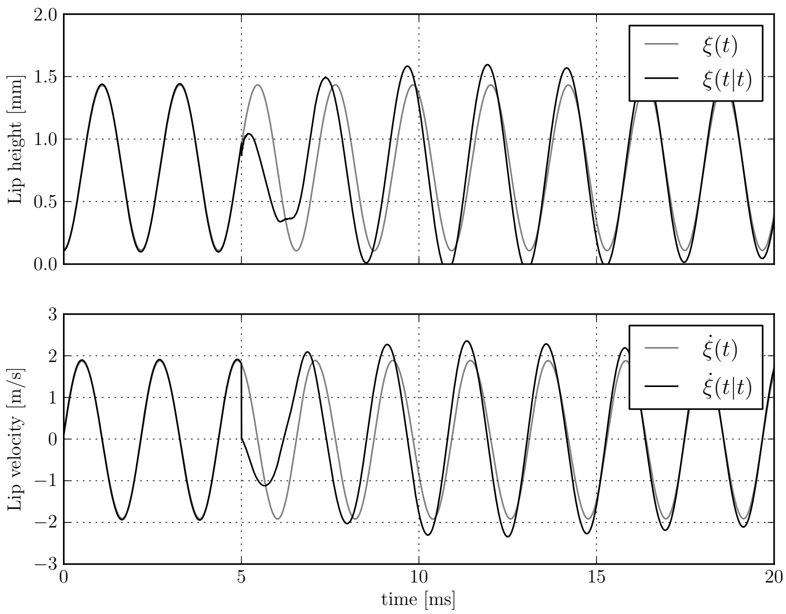

Figure 2: Kalman filter state estimates for ξ and $\dot{\xi}$.

behavior of the filter is depicted on the picture below when $t < 5$ ms, for a signal-to-noise ratio of 12 dB. In this context, the lip height and lip estimate relative error are below 1 %. Note that the direct estimation of the height by sensitivity methods yields at the best of times an error around 10 % and that the maximal error is far larger.

We trigger artificially a reset of the Kalman filter at $t = 5$ ms to display the transitional behavior; a few cycles are enough to provide estimates with an error below 10 % at all instants of the cycle.

Appendix: Sound to Height Sensitivity

Let r be the function of (p_m, y_-, y_+) defined as

$$r := \frac{y_+ - \lambda y_-}{(p_m - y_+) + \lambda(p_m - y_-)}. \tag{36}$$

When $p_m = p_m(t)$, $y_- = y(t - \tau/2)$ and $y_+ = y(t + \tau/2)$, r is adimensional and merely proportional to the ratio between the airflow at the origin of the pipe $\phi(t)$ and the difference of pressure $\Delta p(t) = p_m(t) - p(t)$ between the mouth and the pipe:

$$r(t) = Z_c \frac{\phi(t)}{\Delta p} = \frac{p^+(t) - p^-(t)}{p_m(t) - p^+(t) - p^-(t)} \tag{37}$$

The expression of Ξ^2 given in (14) yields

$$\nabla_y \Xi^2 = \frac{\mu}{2(1+\lambda)} \begin{bmatrix} \lambda r(r-2) \\ r(r+2) \end{bmatrix} \tag{38}$$

The function Ξ^2 itself can be expressed as

$$\Xi^2 = \frac{\mu}{2} r^2 \Delta p. \tag{39}$$

Hence, as $\nabla_y \Xi = \nabla_y \Xi^2 / 2\sqrt{\Xi^2}$, we obtain

$$\nabla_y \Xi = \sqrt{\frac{\mu}{2\Delta p}} \frac{1}{1+\lambda} \begin{bmatrix} \lambda(r/2 - 1) \\ (r/2 + 1) \end{bmatrix} \tag{40}$$

and

$$\|\nabla_y \Xi\| = \sqrt{\frac{\mu}{2\Delta p} \frac{1}{(1+\lambda)^2} \left[\left(\frac{r}{2} + 1\right)^2 + \lambda^2 \left(\frac{r}{2} - 1\right)^2 \right]}. \tag{41}$$

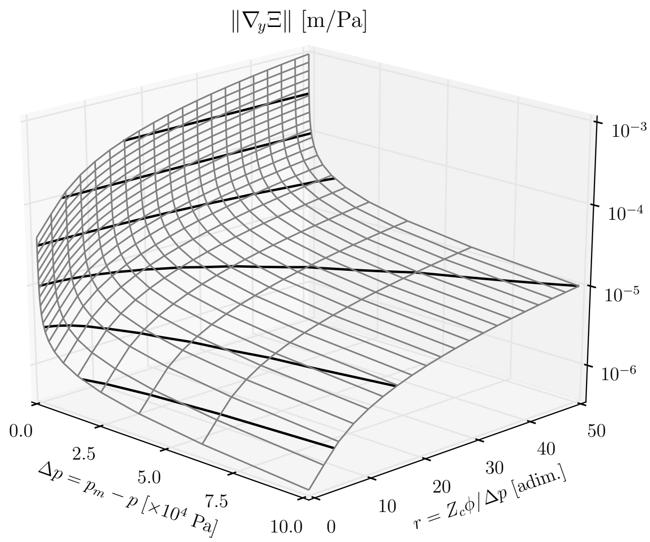

Figure 3: Sound to lip height sensitivity, with iso-value curves in black.

References

Adachi, Seiji, and Masa-aki Sato. 1996. "Trumpet Sound Simulation Using a Two-Dimensional Lip Vibration Model." *Journal of the Acoustical Society of America* 99 (2): 1200–1209. http://profiles.wizfolio.com/loisdankiewicz/publications/2260/104091/.

Bellen, Alfredo, and Marino Zennaro. 2003. *Numerical Methods for Delay Differential Equations.* Oxford: Oxford University Press. doi:10.1093/acprof:oso/9780198506546.001.0001.

Boisgérault, Sébastien. 2013. "Growth Bound of Delay-Differential Algebraic Equations." *Comptes Rendus de L'Académie Des Sciences - Series I - Mathematics* 351 (15-16) (August): 645–648. doi:10.1016/j.crma.2013.08.001 http://hal-ensmp.archives-ouvertes.fr/hal-00877419.

d'Andréa-Novel, Brigitte, Jean-Michel Coron, and Thomas Hélie. 2010. "Asymptotic State Observers for a Simplified Brass Instrument Model." *Acta Acustica* 96-4: 733–742.

Ma, L, H Wang, and J Chen. 2010. "Analysis of Kalman Filter with Correlated Noises Under Different Dependence." *J Inf Comput Sci* 7 (5): 1147–1154.