Elements of Lagrangian Mechanics
Applications to Mobile Robotics

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Modelling of Mechanical Systems – A Short History

- Precursors: Galileo, Descartes, etc.
- **Classical (Newtonian) Mechanics:**
  Newton, 1687: *Philosophiae Naturalis Principia Mathematica*
- **Analytical (Lagrangian) Mechanics:**
  Joseph-Louis Lagrange, 1788: *Méchanique Analitique*
- **Analytical (Hamiltonian) Mechanics:**
  William Rowan Hamilton 1834: *On the Application to Dynamics of a General Mathematical Method previously applied to Optics*

What's new in Analytical Mechanics ?

Same modelling of mechanical systems in the end, but a different way to discover the system equations.

Analytical Mechanics:

- relies on the *principle of stationary action* (calculus of variation, good for maths),
- the steps are easy to automate (good for complex systems and computers),
- describes constrained mechanical systems without extra hassle (good for robotics),
- unveils additional structure of the mechanical system (good for physics and control).

Lagrangian Mechanics Concepts

- $q \in \mathbb{R}^n$: generalized coordinates,
- $\dot{q} \in \mathbb{R}^n$: generalized velocities,
- $K(q, \dot{q}) \in \mathbb{R}$: kinetic energy,
- $V(q) \in \mathbb{R}$: potential energy,
- $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$: **lagrangian**,
- $f \in \mathbb{R}^n$: extra forces.

Euler-Lagrange Equations

The trajectories $q(t)$ followed by the system satisfy:

$$\frac{d}{dt}\nabla_{\dot{q}} L - \nabla_q L = f$$

It's a result of the **principle of stationary action**: we solve

$$\min_q A(q)$$

with

$$A(q) = \int L(q(t), \dot{q}(t)) \, dt$$

Examples - Punctual Mass

A point with mass m at the location (x, y, z), subject to the force f.

$$q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$L(q, \dot{q}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

The Euler-Lagrange equation delivers:

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = f$$

Examples - Simple Pendulum, 1 d.o.f.

Consider the planar system made of a punctual mass m connected by a massless rod of length ℓ to the origin, subject to the gravity force.

We select as a generalized coordinate q the angle θ between the rod and the lowest position. Then, the lagrangian is equal to:

$$L(\theta, \dot{\theta}) = \frac{1}{2} m(\ell\dot{\theta})^2 + mg\ell \cos\theta$$

The Euler-Lagrange equation provides:

$$m\ell^2\ddot{\theta} + mg\ell \sin\theta = 0$$

Examples - Simple Pendulum, 2 d.o.f. (3D)

Same setting, but without the planar restriction.

A new angle α determines the rotation w.r.t. the vertical axis of the rod.

$$q = \begin{bmatrix} \theta \\ \alpha \end{bmatrix}$$

$$L(q, \dot{q}) = \frac{1}{2} m\ell^2 (\dot{\theta}^2 + (\sin\theta)^2 \dot{\alpha}^2) + mg\ell\cos\theta$$

The Euler-Lagrange equation provides:

$$m\ell^2 \begin{bmatrix} 1 & 0 \\ 0 & (\sin\theta)^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = m\ell^2 \cos\theta \sin\theta \begin{bmatrix} \dot{\alpha}^2 \\ -2\dot{\alpha}\dot{\theta} \end{bmatrix} - \begin{bmatrix} mg\ell\sin\theta \\ 0 \end{bmatrix}$$

Euler-Lagrange Equations - Explicit Form

Write

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^t M(q)\dot{q}$$

where $M(q) \in \mathbb{R}^{n\times n}$ is the symmetric, non-negative **(generalized) mass (inertia) matrix**. The Euler-Lagrange equations become

$$M(q)\ddot{q} = f - \nabla_q V(q) - C(q, \dot{q})\dot{q}$$

where $C(q, \dot{q})\dot{q}$ are the centrifugal and Coriolis forces.

$$C(q, \dot{q})\dot{q} = \frac{dM(q)}{dt}\dot{q} - \frac{1}{2}\nabla_q \left(\dot{q}^t M(q)\dot{q}\right)$$

Centrifugal and Coriolis forces

The matrix $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ may be defined in terms of the **Christoffel symbols** of the tensor metric M:

$$C_{ij} = \frac{1}{2} \sum_k \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i} \right) \dot{q}_k$$

N.B.: the matrix

$$\frac{d}{dt} M(q) - 2C(q, \dot{q})\dot{q}$$

is anti-symmetric.

Total Mechanical Energy

The total mechanical energy of the system is

$$H(q, \dot{q}) = K(q, \dot{q}) + V(q)$$

Its evolution is driven by:

$$\dot{H} = f \cdot \dot{q}$$

Hamiltonian Mechanics

It provides an alternate representation of the Euler-Lagrange equations.

Consider L as a function of $\dot{q}$ for a fixed value of q.

Compute its **Legendre transform** $H(p, q)$:

$$H(p, q) = \max_{\dot{q} \in \mathbb{R}^n} \left(p \cdot \dot{q} - L(q, \dot{q}) \right)$$

We call:

- p: **(generalized) momentum**,
- H: **hamiltonian**.

Assume that

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^t M(q)\dot{q},\ \ M(q) = M(q)^t,\ \ M(q) > 0.$$

There is a unique $\dot{q}$ solution of the Legendre maximisation problem, and one-to-one mapping

$$(q,\dot{q}) \longleftrightarrow (p,q).$$

We also have

$$p = \nabla_{\dot{q}} L(q,\dot{q}) = M(q)\dot{q}$$

$$H(p,q) = \frac{1}{2}p^t M(q)^{-1} p + V(q) = K(p,q) + V(q)$$

H is the **total mechanical energy** of the system.

$$\nabla_p H(p,q) = \dot{q} + (\nabla_p \dot{q})p - (\nabla_p \dot{q})\nabla_{\dot{q}} L(q,\dot{q}) = \dot{q}$$

$$\nabla_q H(p,q) = (\nabla_q \dot{q})p - \nabla_q L(q,\dot{q}) - (\nabla_q \dot{q})\nabla_{\dot{q}} L(q,\dot{q})$$

$$\nabla_q H(p,q) = -\nabla_q L(q,\dot{q}) = f - \frac{d}{dt}\nabla_{\dot{q}} L = f - \dot{p}$$

These computations lead to:

$$\left\{ \begin{array}{rcl} \dot{q} & = & +\nabla_p H(p,q) \\ \dot{p} & = & -\nabla_q H(p,q) + f \end{array} \right.$$

It's now trivial to prove results such as:

$$\dot{H} = f \cdot \dot{q}$$

Change of Coordinates

The Euler-Lagrange equations are invariant by change of coordinate system: if

$$\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_{q}L = f$$

and

$$q' = \phi(q)$$

then

$$\frac{d}{dt}\nabla_{\dot{q}'}L - \nabla_{q'}L = f'$$

with

$$\dot{q}' = [\nabla_q \phi(q)]^t \dot{q}$$

$$f' = [\nabla_q \phi(q)]^{-t} f$$

so that the power expression is preserved through the change of variables:

$$P = f \cdot q = f' \cdot q'$$

Constrained Systems

Consider mechanical systems subject to constrained on the admissible motions having the structure

$$\Sigma(q)\dot{q} = 0, \; \Sigma(q) \in \mathbb{R}^{m \times n}$$

Note that geometric – or **holonomic** - constraints, which have the structure $G(q) = 0$, may be described in this framework with

$$\Sigma(q) = \nabla_q G(q)^t.$$

If the extra force f_Σ needed to enforce the constraint does not work (no exchange of energy), we end up with

$$\exists\, \lambda \in \mathbb{R}^m, \; f_\Sigma = \Sigma(q)^t\, \lambda$$

$$\begin{cases} M(q)\ddot{q} &= f - \nabla_q V(q) - C(q,\dot{q})\dot{q} + \Sigma(q)^t \lambda \\ \Sigma(q)\dot{q} &= 0 \end{cases}$$

Assume that $\Sigma(q) \in \mathbb{R}^{m \times n}$ is full rank ($m < n$), that is

$$\dim \ker \Sigma(q) = n - m$$

Pick a $n \times (n-m)$ matrix $S(q)$ of full rank $(n-m)$ such that:

$$\Sigma(q)S(q) = 0$$

$$\Sigma(q)\dot{q} = 0 \iff \exists\, \eta \in \mathbb{R}^{n-m},\ \dot{q} = S(q)\eta$$

The vector η explicits the **degrees of freedom** of the system.

Equivalent to

$$\left\{ \begin{array}{rcl} M_r(q)\dot{\eta} & = & S(q)^t(f - \nabla_q V(q) - C(q,\dot{q})\dot{q} - M(q)\frac{dS(q)}{dt}\eta) \\ \dot{q} & = & S(q)\eta \end{array} \right.$$

with the **reduced mass matrix**:

$$M_r(q) = S(q)^t M(q) S(q)$$

Kinematic Modelling of Mobile Robots

Consider a mobile robots with:

- n – fixed or orientable – wheels,
- a rigid chassis.

We attach to the chassis a mobile frame, described w.r.t. a fixed frame by:

- the coordinates x, y of its origin,
- an orientation angle θ.

We set

$$\xi = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

A wheel i is described by:

- a (common) wheel radius r,
- its coordinates (X_i, Y_i) in the mobile frame,
- the steering angle γ_i,
  (0 when the wheel points in direction of the 2nd frame axis).
- the wheel rotation angle ϕ_i.

Rotation Matrices

Let

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and

$$\mathcal{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R(\theta)$ is a **rotation matrix**,
- $\mathcal{R}(\theta)$ is a **homogeneous rotation matrix**.

No-Slip Pure Roll (NSPR) Condition

The velocity of the center of the wheel i, in the fixed frame coordinates, is given by:

$$\vec{v}_i = R(\theta)R(\gamma_i)\left[\begin{array}{c} 0 \\ r\dot{\phi}_i \end{array}\right]$$

Under the NSPR assumption, it also is:

$$\vec{v}_i = \left[\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right] + \dot{\theta}R(\pi/2)R(\theta)\left[\begin{array}{c} X_i \\ Y_i \end{array}\right]$$

Collecting these equations for all wheels leads to:

$$\left| \begin{array}{rcl} C(\gamma)\mathcal{R}(\theta)^{-1}\dot{\xi} & = & 0 \\ J(\gamma)\mathcal{R}(\theta)^{-1}\dot{\xi} & = & r\dot{\phi} \end{array} \right.$$

with

$$C(\gamma) = \begin{bmatrix} \vdots & \vdots & \vdots \\ +\cos\gamma_i & +\sin\gamma_i & +X_i\sin\gamma_i - Y_i\cos\gamma_i \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$J(\gamma) = \begin{bmatrix} \vdots & \vdots & \vdots \\ -\sin\gamma_i & +\cos\gamma_i & +X_i\cos\gamma_i + Y_i sin\gamma_i \\ \vdots & \vdots & \vdots \end{bmatrix}$$

The kinematic constraint $C(\gamma)\mathcal{R}(\theta)^{-1}\dot{\xi} = 0$ may be explicited if we introduce full-rank matrix $\Sigma(q)$ of size $n \times m$ whose columns form a basis of $\ker C(\gamma)$.

$$C(\gamma)\Sigma(\gamma) = 0$$

The kinematic equations may be rewritten as:

$$\left| \begin{array}{rcl} \dot{\xi} & = & \mathcal{R}(\theta)\Sigma(\gamma)\eta \\ r\dot{\phi} & = & J(\gamma)\Sigma(\gamma)\eta \end{array} \right.$$

where $\eta \in \mathbb{R}^m$ is a free vector.

Geometric Interpretation

There is a unique location where the velocity of the chassis is 0, unless the velocity field is a **uniform translation**.

This is the **instantaneous center of rotation** of the system.

Let $(x^\star, y^\star)$ be its coordinates in the mobile frame. They are solutions of:

$$\left[\begin{array}{c} 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right] + \dot{\theta} R(\theta) R(\pi/2) \left[\begin{array}{c} x^\star \\ y^\star \end{array}\right],$$

that is

$$\left[\begin{array}{c} x^\star \\ y^\star \end{array}\right] = R(\pi/2 - \theta) \left[\begin{array}{c} \dot{x}/\dot{\theta} \\ \dot{y}/\dot{\theta} \end{array}\right]$$

Homogeneous coordinates $(X^\star, Y^\star, Z^\star)$ of the ICR is a triple such that there is a $t \in \mathbb{R}$ with

$$\begin{bmatrix} X^\star \\ Y^\star \\ Z^\star \end{bmatrix} = \begin{bmatrix} tx^\star \\ ty^\star \\ t \end{bmatrix}$$

A triple of homogeneous coordinates for the ICR are obtained by:

$$\begin{bmatrix} X^\star \\ Y^\star \\ Z^\star \end{bmatrix} = \mathcal{R}(\pi/2)\mathcal{R}(\theta)^{-1}\dot{\xi}$$

It is such that $Z^\star = \dot{\theta}$.

The uniform translation may be described in the same setting:

it corresponds to $Z^\star = \dot{\theta} = 0$.

The kinematic constraints on $\dot{\xi}$ given by

$$C(\gamma)\mathcal{R}(\theta)^{-1}\dot{\xi} = 0$$

are equivalent to the constraints

$$C(\gamma)\mathcal{R}(-\pi/2)\begin{bmatrix} X^\star \\ Y^\star \\ Z^\star \end{bmatrix} = 0$$

Example – Kinematic Chariot Model

Select a mobile frame in the middle of the two wheels, the second axis pointing forward:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} -e \\ 0 \end{bmatrix}, \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} +e \\ 0 \end{bmatrix}$$

Fixed wheel orientations $\gamma_1 = \gamma_2 = 0$.

$$C(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ J(\gamma) = \begin{bmatrix} 0 & 1 & -e \\ 0 & 1 & +e \end{bmatrix}$$

We have $\dim \ker C(\gamma) = 2$. We may pick

$$\Sigma(\gamma) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \eta \in \mathbb{R}^2$$

That choice leads to:

$$\left|\begin{array}{rcl} \dot{x} & = & -(\sin\theta)\eta_1 \\ \dot{y} & = & +(\cos\theta)\eta_1 \\ \dot{\theta} & = & \eta_2 \\ r\dot{\phi}_1 & = & \eta_1 - e \times \eta_2 \\ r\dot{\phi}_2 & = & \eta_1 + e \times \eta_2 \end{array}\right.$$

Hence:

- η_1 is the linear velocity of the center of the mobile frame,
- η_2 is the chassis angular velocity.

Example – Kinematic Bicycle Model

Select a mobile frame centered on the fixed wheel, the second axis pointing forward:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \ell \end{bmatrix}$$

Fixed wheel $\gamma_1 = 0$, orientable wheel γ_2.

$$C(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ \cos\gamma_2 & \sin\gamma_2 & -\ell\cos\gamma_2 \end{bmatrix}$$

$$J(\gamma) = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\gamma_2 & \cos\gamma_2 & \ell\sin\gamma_2 \end{bmatrix}$$

We have $\dim\ker C(\gamma) = 1$. We may pick

$$\Sigma(\gamma) = \begin{bmatrix} 0 \\ \cos\gamma_2 \\ \sin\gamma_2/\ell \end{bmatrix}, \ \eta \in \mathbb{R}^1$$

That choice leads to:

$$\left|\begin{array}{rcl} \dot{x} & = & -(\sin\theta\cos\gamma_2)\eta_1 \\ \dot{y} & = & +(\cos\theta\cos\gamma_2)\eta_1 \\ \dot{\theta} & = & (\sin\gamma_2/\ell)\eta_1 \\ r\dot{\phi}_1 & = & (\cos\gamma_2)\eta_1 \\ r\dot{\phi}_2 & = & \eta_1 \end{array}\right.$$

Hence:

- η_1 is the linear velocity of the front wheel.

Example - Dynamic Chariot Model

Kinetic Energy: if the mass M of the chassis is uniformly distributed on a disk of radius r centered in the middle of the wheels, the mass density ρ satisfies:

$$\rho(x, y) = \left| \begin{array}{cl} M/(\pi e^2) & \text{if } x^2 + y^2 \leq e^2, \\ 0 & \text{otherwise.} \end{array} \right.$$

The kinetic energy is

$$K = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$$

where the **moment of inertia** is given by:

$$I = \int \rho(x, y)(x^2 + y^2) dxdy = \frac{Me^2}{2}.$$

The **mass matrix** of the (reduced) system with generalized coordinates $q = \xi$ is:

$$M(\xi) = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & Me^2/2 \end{bmatrix}$$

This matrix is independant of ξ, therefore the Coriolis and centrifugal forces are 0. Moreover

$$S(\xi) = \begin{bmatrix} -\sin\theta & 0 \\ +\cos\theta & 0 \\ 0 & 1 \end{bmatrix}$$

$$-S(\xi)^t M(\xi) \frac{dS(\xi)}{dt} = 0$$

$$M_r(\xi) = S(\xi)^t M(\xi) S(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & Me^2/2 \end{bmatrix}$$

The generalized (motor) forces f that correspond to the generalized coordinates $\xi = (x, y, \theta)$ are:

$$f = \begin{bmatrix} f_x \\ f_y \\ c \end{bmatrix}$$

where (fx, fy) are the cartesian coordinates of the motor force exerted on the center of the mobile frame and c is the torque applied to the chassis.

Consequently, the kinematic equations are supplemented by:

$$\left| \begin{array}{rcl} M \times \dot{\eta_1} & = & -f_x \sin\theta + f_y \cos\theta \\ Me^2/2 \times \dot{\eta_2} & = & c \end{array} \right.$$

Structural Properties - Typology of Mobile Robots

The type of a wheeled mobile robot is a pair (δ_m, δ_s) where:

- δ_m is the **degree of mobility**: the number of degrees of freedom in a given wheel configuration:

$$\delta_m = \dim \ker C(\gamma) = \dim\{\text{admissible ICR}\} + 1$$

- δ_s is the **degree of steerability**:
  the extra number of degrees of freedom associated to orientable wheels.

Robots with the same type share similar structural properties.