Shape derivative of sharp functionals governed by Navier-Stokes flow

Sébastien Boisgérault*, Jean-Paul Zolésio*

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Abstract: The shape analysis of the Navier-Stokes equation has been already considered in the literature. Classical techniques, such as the Implicit Function Theorem, may be used to show that some functionals, the drag for example, are shape differentiable.

However, this property relies on results established for the basic regularity of the pressure and the velocity fields. Many other criterions of physical interest are out of this scope: we consider here the shape analysis of such functionals, for example, the (total) force exerted by the fluid on a body or the moment of these forces. The velocity and pressure fields u and p are assumed to be solutions of the stationary incompressible Navier-Stokes equation $-\nu\Delta u + [\mathrm{D}u]u + \nabla p = f$ in Ω with the boundary condition $u|_\Gamma = 0$, $\Gamma = \partial\Omega$.

These new results are based on the so-called speed method which allows us to "bring back" vector fields from a perturbed domain to the initial one while preserving the divergence-free property. Regularity results are established for that correspondence and used to define and show some properties of the shape derivative u' and of the boundary shape derivative u'_Γ.

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1 Introduction

We study the shape differentability of functionals of the solutions (u, p) of the Navier-Stokes Equation in Ω_0 which are not defined for the minimal regularity of the solutions,

*CMA, Ecole des Mines de Paris – INRIA, 2004 route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex, France. E-mail: *Sebastien.Boisgerault@sophia.inria.fr* and *Jean-Paul.Zolesio@sophia.inria.fr*. Fax: +33 4 92 38 79 98.

for example because high-order derivatives or integrals on submanifolds appear in the expression of these functionals.

In the framework of the Speed Method (section 3.1), some perturbed sets $[s \mapsto \Omega_s]$ are associated to the initial set Ω_0, the velocities u_s solutions of the Navier-Stokes problem in the moving sets Ω_s are associated to fields u^s in the fixed set Ω_0 (section 3.2) which are solutions of the *Transported Navier-Stokes Equation* (section 3.3). The definition of this equation uses the functional spaces introduced in the section 2. The Implicit Function Theorem is then used to prove some regularity of $[s \mapsto u^s]$ for the desired spatial regularity of u (section 3.4). This result allows us to show the existence of the shape derivatives of u and p for a given spatial regularity and then, through the development of a Tangential Calculus, their boundary shape derivatives (section 4). Theses objects are then intensively used to obtain the explicit form of the shape gradient of the force and the moment of the forces applied by a fluid on a part of its boundary (section 5).

2 The Navier-Stokes Equations

The Navier-Stokes problem in an open bounded set $\Omega \subset \mathbb{R}^3$ is classically written as

$$\left\{ \begin{array}{ccl} -\nu\Delta u + [\mathrm{D}u]u + \nabla p = f & \text{in } \Omega & (1.1) \\ \operatorname{div} u = 0 & \text{in } \Omega & (1.2) \\ u = 0 & \text{on } \partial\Omega & (1.3) \end{array} \right. \tag{1}$$

where ν is the kinematic viscosity, u the velocity of the fluid, p the pressure and f the force. The study of this equation requires functional spaces built as subspaces (to deal with the fluid incompressibility (1.2) or the boundary value (1.3)) or quotients of Sobolev spaces (to handle forces defined up to a gradient): for any integer $m \geq 1$, we define

$$V^m(\Omega) = \{u \in H^m(\Omega;\mathbb{R}^3) \cap H^1_0(\Omega;\mathbb{R}^3),\ \operatorname{div} u = 0\} \tag{2}$$

endowed with the $H^m(\Omega;\mathbb{R}^3)$ norm and for any integer $m \geq -1$, we set

$$W^m(\Omega) = H^m(\Omega;\mathbb{R}^3)/\{\nabla p,\ p \in H^{m+1}(\Omega;\mathbb{R})\} \tag{3}$$

endowed with the quotient norm. The linear continuous mappings

$$i : V^m(\Omega) \to H^m(\Omega;\mathbb{R}^3) \quad \text{and} \quad \pi : H^m(\Omega;\mathbb{R}^3) \to W^m(\Omega) \tag{4}$$

are respectively the canonical injection and the quotient mapping.

Remark 1 Strictly speaking, we did not define uniquely the mappings i and π, but only some sequences $(i_m)_{m\geq 1}$ and $(\pi_m)_{m\geq -1}$. It is obvious that when $u \in H^m(\Omega)$, for any integer m', $1 \leq m' \leq m$, $i_m(u) = i_{m'}(u)$, so i is uniquely determined. For π, we must first notice that for any $-1 \leq m' \leq m$, the mapping $\pi_m(f) \in W^m(\Omega) \mapsto \pi_{m'}(f) \in W^{m'}(\Omega)$ is well defined (that is, does not depends on the choice of f) and is a linear continuous injection, so we may identify $W^m(\Omega)$ with a subspace of $W^{m'}(\Omega)$. With this convention, $\pi_m(f) = \pi_{m'}(f)$ when the two expressions make sense, and π is also well defined. Moreover, $\pi(f) \in W^m(\Omega)$ if and only if f may be decomposed as $f = g + \nabla p$, $g \in H^m(\Omega;\mathbb{R}^3)$ and $p \in L^2(\Omega;\mathbb{R})$ (and not necessarily $p \in H^m(\Omega;\mathbb{R})$): the spaces $W^m(\Omega)$ characterize the best possible regularity of f "up to a gradient" (whatever its regularity is).

Let $\mathcal{D}(\Omega;\mathbb{R}^3)$ be the set of functions of $C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ whose support is compactly included in Ω. We recall that if Ω is Lipschitz, $\mathcal{V}(\Omega) = \{v \in \mathcal{D}(\Omega;\mathbb{R}^3), \operatorname{div} v = 0\}$ is dense in $V^1(\Omega)$. Consequently,

Proposition 1 *The spaces $W^{-1}(\Omega)$ and $V^1(\Omega)'$ are isomorphic. Moreover,*

$$\begin{aligned} \forall f \in H^{-1}(\Omega;\mathbb{R}^3) \qquad \pi(f) = 0 &\iff \forall v \in V^1(\Omega),\ < f, v >= 0 \\ &\iff \forall v \in \mathcal{V}(\Omega),\ < f, v >= 0 \end{aligned} \tag{5}$$

Proof: The linear continuous operator $f \in H^{-1}(\Omega;\mathbb{R}^3) \mapsto f|_{V^1(\Omega)} \in V^1(\Omega)'$ is clearly onto and $f \in H^{-1}(\Omega;\mathbb{R}^3)$ belongs its kernel iff

$$\forall v \in V^1(\Omega),\ < f, v >= 0 \iff \exists p \in L^2(\Omega;\mathbb{R}),\ f = \nabla p \iff \pi(f) = 0$$

(see [10] for the first equivalence), which proves the first part of (5), and that $V^1(\Omega)' \simeq H^{-1}(\Omega;\mathbb{R}^3)/\{\nabla p,\ p \in L^2(\Omega;\mathbb{R})\} = W^{-1}(\Omega)$. The second equivalence follows by density. ■

The operators π and i will be used to show some regularity results for some (one-variable or two-variable) mappings from $V^m(\Omega)$ to $W^{m'}(\Omega)$ which are build on mappings from $H^m(\Omega;\mathbb{R}^3)$ to $H^{m'}(\Omega;\mathbb{R}^3)$ (as it is shown in the following definition) whose regularity is known.

Definition 1 *For any integer n and mapping $T : (H^1(\Omega;\mathbb{R}^3))^n \to H^{-1}(\Omega;\mathbb{R}^3)$, we define the mapping $\bar{T} : (V^1(\Omega))^n \to W^{-1}(\Omega)$ by*

$$\bar{T} = \pi \circ T \circ (i \otimes i \otimes ... \otimes i) \tag{6}$$

In particular, this construction is applied for the operators A and B defined by

$$< Au, \varphi >_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega \mathrm{D}u \cdot\cdot \mathrm{D}\varphi \, dx, \; < B(u,v), \varphi >_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega [\mathrm{D}u]v \cdot \varphi \, dx \tag{7}$$

for $u \in H^1_{\mathrm{loc}}(\Omega)$ and $v \in L^2_{\mathrm{loc}}(\Omega)$. These operators, used in the variational formulation of (1), have the following regularity :

Proposition 2 *For any integer $n \geq 1$, A is a continuous mapping $H^n(\Omega;\mathbb{R}^3) \to H^{n-2}(\Omega;\mathbb{R}^3)$. B is a continuous mapping $H^1(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^3) \to H^{-1}(\Omega;\mathbb{R}^3)$ and $H^n(\Omega;\mathbb{R}^3) \times H^n(\Omega;\mathbb{R}^3) \to H^{n-1}(\Omega;\mathbb{R}^3)$ for any integer $n \geq 2$.*

Sketch of the proof: The regularity of A is classical. The one of B is a consequence of the continuity of the trilinear form

$$(u,v,w) \in H^a(\Omega;\mathbb{R}) \times H^b(\Omega;\mathbb{R}) \times H^c(\Omega;\mathbb{R}) \mapsto \int_\Omega uvw \, dx$$

for nonnegative real numbers a,b,c such that $a+b+c > 3/2$ (see [4]). ■

The proposition 1 shows that the usual variational formulation of the Navier-Stokes problem is equivalent to the equation

$$\nu \bar{A}u + \bar{B}(u,u) = \pi(f) \tag{8}$$

Moreover, the regularity of the Stokes equation combined with iterated evaluations of the regularity of $[\mathrm{D}u]u$ shows that if Ω is of class C^r, $r = \max(2, m+2)$, and $f \in H^m(\Omega;\mathbb{R}^3)$, $m \geq -1$, any solution u of the Navier-Stokes equation is in $H^{m+2}(\Omega;\mathbb{R}^3)$.

3 Transport

3.1 The Speed Method

In this section, we consider a *hold-all* D which contains the set Ω_0 filled by the fluid and a (time-dependent) vector field V defined on D which is used to define the family of perturbed domain Ω_s based on Ω_0: each point $x \in \Omega_0$ is continuously transported by the ODE defined by the field V. The parameter which controls the amplitude of the deformation is denoted by s.

Precisely, the hold-all D is assumed to be a domain at least of class C^k, $k \geq 1$ and s is in a (possibly infinite) interval $I \subset \mathbb{R}$ such that $0 \in I$. The vector field V is assumed to be an element of the set $E^{n,k}(I, D)$ (or simply $E^{n,k}$) defined by:

$$E^{n,k}(I, D) = \left\{ V \in C^n(I; C^k(\overline{D}; \mathbb{R}^3)) \mid \forall s \in I,\ V(s) \cdot n = 0 \text{ on } \partial D \right\} \tag{9}$$

where n denotes the unitary outer normal to D.

Then we may define the mapping $s \mapsto T_s(V)$ (or simply T_s when there is no possible confusion on the vector field) as the solution of

$$\forall s \in I,\ \frac{dT_s}{ds} = V(s) \circ T_s, \quad \text{and} \quad T_0 = \mathrm{Id} \tag{10}$$

and also the family of perturbed sets $s \mapsto \Omega_s$ by

$$\Omega_s = T_s(\Omega_0) \tag{11}$$

We recall the following result which can be found in [9], [11]:

Proposition 3 *Let $V \in E^{n,k}(I, D)$, $n \geq 0$, $k \geq 1$, be a given vector field. Then*

i) *$\forall s \in I$, $T_s(V) : D \to D$ and $\overline{D} \to \overline{D}$ are one-to-one mappings.*

ii) *$s \mapsto T_s(V) \in C^{n+1}(I; C^k(\overline{D}; \overline{D}))$ and $s \mapsto [T_s(V)]^{-1} \in C^n(I; C^k(\overline{D}; \overline{D}))$.*

iii) *$\forall s \in I$, $\forall x \in \overline{D}$, $\mathrm{D}T_s(x)$ is invertible, and the mappings $s \mapsto \mathrm{D}T_s$ and $s \mapsto [\mathrm{D}T_s]^{-1}$ are in $C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}^{3\times 3}))$.*

As a first consequence of this proposition, the family of perturbed sets has its boundary regularity preserved for V smooth enough: if Ω_0 is of class C^r, $r \leq k$, then for any $s \in I$, Ω_s is also of class C^r.

Remark 2 In order to make the future calculations easier, we define

$$\gamma_s = \det(\mathrm{D}T_s) \text{ and } C_s = \gamma_s^{-1}\mathrm{D}T_s \tag{12}$$

Notice that, as a simple consequence of the Proposition 3, for $V \in E^{n,k}$, $n \geq 0$, $k \geq 1$, $\forall s \in I$, $\gamma_s^{-1} = \det([\mathrm{D}T_s]^{-1})$ exists and moreover, γ_s and $\gamma_s^{-1} \in C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}^{3\times 3}))$. As $\gamma_0 = 1$, by continuity, $\forall s \in I$, $\gamma_s > 0$. Obviously, we also have the inversibility of C_s for any $s \in I$ and the regularity C_s and $C_s^{-1} \in C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}^{3\times 3}))$.

3.2 Correspondence between vector fields

The diffeomorphisms T_s defined in the previous section are used to build a correspondence between the mappings defined on Ω_0 and those defined on Ω_s. As for any $s \in I$, T_s and T_s^{-1} are Lipschitz, the mapping $[v \mapsto v \circ T_s]$ is an isomorphism between $H_0^1(\Omega_s; \mathbb{R})$ and $H_0^1(\Omega_0; \mathbb{R})$ (see [8]) and the equation $\mathrm{D}[v \circ T_s] = [\mathrm{D}v \circ T_s]\mathrm{D}T_s$ holds.

Lemma 1 *Assume that Ω_0 is an open bounded subset of $\mathbb{R}^3$ and that $V \in E^{0,2}$. Then the mapping $\mathbb{T}_s$, defined between functions $\Omega_0 \to \mathbb{R}^3$ and $\Omega_s \to \mathbb{R}^3$ by*

$$\mathbb{T}_s(u) \circ T_s = C_s\, u \tag{13}$$

is an isomorphism between $V^1(\Omega)$ and $V^1(\Omega_s)$.

Proof: As for all $s \in I$, C_s and C_s^{-1} are $C^1(\overline{D}; \mathbb{R}^{3\times 3})$ (see remark 2), $\mathbb{T}_s$ is an isomorphism between $H_0^1(\Omega_0; \mathbb{R}^3)$ and $H_0^1(\Omega_s; \mathbb{R}^3)$. Moreover, for any $u \in H_0^1(\Omega_0; \mathbb{R}^3)$ and $\varphi \in H_0^1(\Omega_s; \mathbb{R}^3)$, we have

$$\int_{\Omega_s} \mathrm{div}\,(\mathbb{T}_s(u))\,\varphi\,dx = -\int_{\Omega_s} \mathbb{T}_s(u) \cdot \nabla\varphi\,dx = -\int_{\Omega_0} (\mathbb{T}_s(u) \circ T_s) \cdot (\nabla\varphi \circ T_s)\,\gamma_s\,dx$$

and as $\nabla(\varphi \circ T_s) = DT_s^*(\nabla\varphi \circ T_s)$,

$$\int_{\Omega_s} \mathrm{div}\,(\mathbb{T}_s(u))\,\varphi\,dx = -\int_{\Omega_0} (C_s^{-1}\mathbb{T}_s(u) \circ T_s) \cdot \nabla(\varphi \circ T_s)\,dx = \int_{\Omega_0} (\mathrm{div}\,u)\,\varphi \circ T_s\,dx = 0$$

The mapping $\varphi \mapsto \varphi \circ T_s$ being an isomorphism between $H_0^1(\Omega_s; \mathbb{R}^3)$ and $H_0^1(\Omega_0; \mathbb{R}^3)$, $\mathrm{div}\,\mathbb{T}_s(u)$ if and only if $\mathrm{div}\,u = 0$ which achieves the proof. ■

Remark 3 As $\mathbb{T}_s : H_0^1(\Omega_0; \mathbb{R}^3) \to H_0^1(\Omega_s; \mathbb{R}^3)$ is an isomorphism, so is its adjoint $\mathbb{T}_s^\star : H^{-1}(\Omega_s; \mathbb{R}^3) \to H^{-1}(\Omega_0; \mathbb{R}^3)$.

3.3 The Transported Navier-Stokes equations

In the sequel, the objects (operators, vector fields, duality brackets) associated to the perturbed set Ω_s will be noted with a subscript s. Many of them are used to defined objects (by different means) associated to the initial set Ω_0 ; they are noted with the superscript s. Precisely, we define the following correspondences:

• Velocity fields $u_s \in V^1(\Omega_s)$ (solutions of the Navier-Stokes equation in Ω_s) and test functions $v_s \in V^1(\Omega_s)$ are associated to the fields $u^s \in V^1(\Omega_0)$ (resp. $v^s \in V^1(\Omega_0)$) by :

$$u^s = \mathbb{T}_s^{-1}(u_s) \text{ (resp. } v^s = \mathbb{T}_s^{-1}(v_s))$$

• Force fields $f_s \in H^{-1}(\Omega_s;\mathbb{R}^3)$ (or in $H^{-1}(D;\mathbb{R}^3)$) are transported in $f^s \in H^{-1}(\Omega_0;\mathbb{R}^3)$ (or in $H^{-1}(D;\mathbb{R}^3)$) defined by

$$f^s = \mathbb{T}_s^\star(f_s)$$

• The (linear and bilinear) operators A_s and B_s from $H^1(\Omega_s;\mathbb{R}^3)$ to $H^{-1}(\Omega_s;\mathbb{R}^3)$ defined by (7) are associated to A^s and B^s linear and bilinear from $H^1(\Omega_0;\mathbb{R}^3)$ to $H^{-1}(\Omega_0;\mathbb{R}^3)$ by:

$$A^s = \mathbb{T}_s^\star \circ A \circ \mathbb{T}_s \text{ and } B^s = \mathbb{T}_s^\star \circ B \circ (\mathbb{T}_s \otimes \mathbb{T}_s) \tag{14}$$

With these notations, we have

Theorem 1 *Assume that $V \in E^{0,k}$, $k \geq 2$ and that $f \in H^{-1}(D;\mathbb{R}^3)$. The field $u_s \in V^1(\Omega_s)$ is a solution of the Navier-Stokes equation in Ω_s if and only if $u^s \in V^1(\Omega_0)$ is a solution of*

$$\nu \bar{A}^s u^s + \bar{B}^s(u^s, u^s) = \pi(f^s) \tag{15}$$

The operators A^s and B^s satisfy for any u, v in $H^1(\Omega_0;\mathbb{R}^3)$

$$A^s u = C_s^\star \mathrm{div}\left(\mathrm{D}[C_s u]\gamma_s^{-1} C_s^{-1} (C_s^\star)^{-1}\right) \text{ and } B^s(u,v) = C_s^\star \mathrm{D}(C_s u)\cdot v \tag{16}$$

and for $f \in L^2(D;\mathbb{R}^3)$, we have

$$f^s = [\mathrm{D}T_s]^*(f \circ T_s) \tag{17}$$

Remark 4 The pressure is not explicitly transported in the theorem 1. Nevertheless, if $p_s \in L^2(\Omega_s;\mathbb{R})/\mathbb{R}$ is a pressure solution of the Navier-Stokes problem in Ω_s, that is if $\nu A_s u_s + B_s(u_s,u_s) + \nabla p_s = f$ then it can be verified that

$$\nu A^s u^s + B^s(u^s,u^s) + \nabla(p_s \circ T_s) = f^s$$

It is therefore natural to define the transported pressure p^s by

$$p^s = p_s \circ T_s \tag{18}$$

The operators involved in the equation 16 have the following regularity:

Proposition 4 *Assume that $V \in E^{0,m+1}$, $m \geq 1$. Then*

i) *$\forall s \in I$, A^s (resp. B^s) are linear (resp. bilinear) continuous from $H^m(\Omega_0;\mathbb{R}^3)$ to $H^{m-2}(\Omega_0;\mathbb{R}^3)$.*

ii) *$[s \mapsto A^s]$ and $[s \mapsto B^s]$ are continuously differentiable in these spaces.*

Proof: We know that for any integer n, $H^n(\Omega_0;\mathbb{R})$ is a $C^n(\overline{\Omega_0};\mathbb{R})$-topological module, so thanks to the regularity of $[s \mapsto \gamma_s]$, $[s \mapsto \gamma_s^{-1}]$, $[s \mapsto C_s]$ and $[s \mapsto C_s^{-1}]$ (see remark 2), we may consider the continuous mappings $\Lambda_s^1 : H^m(\Omega_0;\mathbb{R}^3) \to H^m(\Omega_0;\mathbb{R}^3)$, $\Lambda_s^2 : H^{m-1}(\Omega_0;\mathbb{R}^{3\times 3}) \to H^{m-1}(\Omega_0;\mathbb{R}^{3\times 3})$ and $\Lambda_s^3 : H^{m-2}(\Omega_0;\mathbb{R}^3) \to H^{m-2}(\Omega_0;\mathbb{R}^3)$, defined by $\Lambda_s^1(u) = C_s u$, $\Lambda_s^2(U) = U\gamma_s^{-1}C_s^{-1}(C_s^\star)^{-1}$ and $\Lambda_s^3(u) = C_s^\star u$.

As the operators A^s and B^s may be decomposed as sequences of linear (or bilinear) continuous operators

$$A^s = \Lambda_s^3 \circ \mathrm{div} \circ \Lambda_s^2 \circ \mathrm{D} \circ \Lambda_s^1 \;\text{ and }\; B^s = \Lambda_s^3 \circ B \circ (\Lambda_s^1 \otimes \mathrm{Id})$$

the property i) is proved. Moreover, as the mappings $[s \mapsto \Lambda_s^i]$ are of class C^1 on I, ii) follows. ■

3.4 Regularity of $[s \mapsto u^s]$

Under the assumptions **A1** and **A2** described below, the theorem 2 characterize the regularity of $[s \mapsto u^s]$.

Assumption 1 (Regularity of the data) *Let $m \geq 2$ be a given integer.*

- *The initial set Ω_0 is an open connected subset included in the hold-all $D \subset \mathbb{R}^3$. The sets Ω_0 and D are respectively of class C^{m+1} and C^{m+2}.*
- *The fields V considered in the speed method are in $E^{1,m+2}(I, D)$.*
- *The force field f is in $H^{m-1}(D; \mathbb{R}^3)$.*

Assumption 2 (Nonsingularity of u) *The field u is a solution of the Navier-Stokes equation in Ω_0 such that the linearized Navier-Stokes equation*

$$-\nu \Delta v + [\mathrm{D}u]v + [\mathrm{D}v]u + \nabla q = g \tag{19}$$

has a unique solution $v \in V^1(\Omega_0)$ for any $g \in H^{-1}(\Omega_0; \mathbb{R}^3)$. Equivalently, the linear operator L from $V^1(\Omega_0)$ to $W^{-1}(\Omega_0)$ defined by $L(v) = \nu \bar{A} + \bar{B}(u, v) + \bar{B}(v, u)$ is an isomorphism.

In particular this assumption is satisfied for high viscosities or small force fields : for a given Ω_0, there is a $k(\Omega_0) > 0$ such that with $\nu^2 / \|f\|_{H^{-1}(\Omega_0;\mathbb{R}^3)} > k(\Omega_0)$ this assumption hold (see [5, lemma 3.2, p. 300]). This threshold also ensures the unicity of the solution of the Navier-Stokes equation in Ω_0.

Theorem 2 *Assume that the assumptions* **A1** *and* **A2** *hold for a given solution u. Then there is on a neighbourhood $J \subset I$ of 0 a unique solution $u_s = u(\Omega_s)$ of the Navier-Stokes problem in Ω_s such that*

i) $u_0 = u$. ii) $[s \mapsto u^s] \in C^0(J; V^{m+1}(\Omega_0)) \cap C^1(J; V^m(\Omega_0))$.

Proof of the theorem 2: Thanks to the theorem 1, u_s is a solution of the Navier-Stokes equation in Ω_s iff u^s is a solution of $\bar{\phi}_s(u) = 0$ where

$$\begin{array}{rccc} \phi_s : & H^m(\Omega_0; \mathbb{R}^3) & \rightarrow & H^{m-2}(\Omega_0; \mathbb{R}^3) \\ & u & \mapsto & A^s u + B^s(u, u) - f^s \end{array}$$

The result of the theorem is proved by two different versions of the Implicit Function Theorem for the same mapping but in different spaces: the mapping $(s, u) \mapsto \bar{\phi}_s(u)$ is considered as an application from $I \times V^{m+1}(\Omega_0)$ to $W^{m-1}(\Omega_0)$ and then stronger properties are exhibited for this mapping defined from $I \times V^m(\Omega_0)$ to $W^{m-2}(\Omega_0)$.

These properties comes from the results already shown for the operators A^s and B^s and from the regularity

$$[s \mapsto f \circ T_s] \in C^0(I; H^{m-1}(D; \mathbb{R}^3)) \cap C^1(I; H^{m-2}(D; \mathbb{R}^3))$$

which hold under the assumption **A1** (see [9]). Precisely, it holds

- The solution u of the Navier-Stokes problem in Ω_0 is in $V^{m+1}(\Omega_0)$ (see section 2) and *a fortiori* in $V^m(\Omega_0)$. It satisfies $\bar{\phi}_0(u) = 0$
- The mapping $(s, u) \to \phi_s(u)$ is in $C^0(I \times H^{m+1}(\Omega_0); H^{m-1}(\Omega_0))$: $(s, u) \to A^s u + B^s(u, u)$ is in fact C^1 in these spaces (see proposition 4) and $[s \mapsto f^s]$ is in $C^0(I; H^{m-1}(\Omega_0; \mathbb{R}^3))$.
- The real $s \in I$ being fixed, for any $m \geq 1$, $u \mapsto \phi_s(u)$ is the sum of a linear continuous, a bilinear continuous and a constant mapping $H^{m+1}(\Omega_0; \mathbb{R}^3) \to H^{m-1}(\Omega_0; \mathbb{R}^3)$ and $H^m(\Omega_0; \mathbb{R}^3) \to H^{m-2}(\Omega_0; \mathbb{R}^3)$ (see proposition 4); $u \mapsto \phi_s(u)$ is therefore a $C^\infty(V^m(\Omega_0); W^{m-2}(\Omega_0))$ and $C^\infty(V^{m+1}(\Omega_0); W^{m-1}(\Omega_0))$ mapping. Consequently, $u \mapsto \bar{\phi}_s(u)$ as the same regularity (in both spaces).
- The proposition 4 also implies that for any $m \geq 1$ and a fixed $u \in V^m(\Omega_0)$, $[s \mapsto A^s u + B^s(u, u)]$ is in $C^1(I; H^{m-2}(\Omega_0))$. The same regularity also for $[s \mapsto f^s]$.
- As $\bar{\phi}_0 = \pi \circ \phi_0 \circ i$, we have $\partial_u \bar{\phi}_0(u)(v) = (\pi \circ \partial_u \phi_0(u) \circ i)(v)$ for any $u \in V^{m+1}(\Omega_0)$ (resp. $u \in V^m(\Omega_0)$). As $\partial_u \phi_0(u)(v) = Av + B(u, v) + B(v, u)$, $\partial_u \bar{\phi}_0(u) = L$, which is an isomorphism from $V^1(\Omega_0)$ to $W^{-1}(\Omega_0)$ (assumption 2). Repeated evaluations of the regularity of the bilinear terms show that in fact the solution v of $L(v) = g$ is in $V^{m+1}(\Omega_0)$ (resp. $V^m(\Omega_0)$) when $g \in H^{m-1}(\Omega_0; \mathbb{R}^3)$ (resp. $g \in H^{m-2}(\Omega_0; \mathbb{R}^3)$). ■

4 The Shape Derivative u' and p'

From now on, we assume that the data are chosen such that **A1** and **A2** are satisfied for an given integer $m \geq 2$. Consequently, the theorem 2 may be applied which allows to show that the solution u and p of the Navier-Stokes problem are shape differentiable in H^m and H^{m-1} (see theorem 3) and that their shape derivative u' and p' are solutions of a well-posed problem (see section 4.4).

4.1 Definition and basic properties

In this section, we recall the basic facts about the shape derivative of a mapping $\Omega \mapsto y(\Omega)$, defined for a given regularity of Ω. y is assumed to be a scalar field for a simplified exposition, but the adaptation to the vectorial case is obvious.

We say that y is *shape differentiable in H^n (resp. C^n) at Ω_0* if for any field V of a given regularity,

i) $[s \mapsto y(\Omega_s) \circ T_s]$ is differentiable in $H^n(\Omega; \mathbb{R})$ (resp. $C^n(\overline{\Omega}; \mathbb{R})$) at $s = 0$. Its derivative, noted $\dot{y}(\Omega)$ or simply $\dot{y}$ is the *material derivative.*

ii) $y(\Omega) \in H^{n+1}(\Omega; \mathbb{R})$ (resp. $y \in C^{n+1}(\overline{\Omega}; \mathbb{R})$).

and the *shape derivative* $y' \in H^n(\Omega; \mathbb{R})$ (resp. $C^n(\overline{\Omega}; \mathbb{R})$) is given by:

$$y' = \dot{y} - \nabla y \cdot V(0) \tag{20}$$

In particular, i) and ii) hold when y satisfies the stronger assumption

iii) $[s \mapsto y(\Omega_s) \circ T_s] \in C^0(I; H^{n+1}(\Omega; \mathbb{R})) \cap C^1(I; H^n(\Omega; \mathbb{R}))$

on a neighbourghood I of 0. If Ω is at least C^{n+1}, we may recover y' as the derivative of an extension of $[s \mapsto y(\Omega_s)]$: we consider an linear extension $P : L^2(\Omega; \mathbb{R}) \to L^2(D; \mathbb{R})$ such that for any integer $r \leq n + 1$, $H^r(\Omega; \mathbb{R})$ is mapped continuously into $H^r(D; \mathbb{R})$. We may then associate to the family $[s \mapsto y(\Omega_s)]$, the family $[s \mapsto Y_s]$ defined on the hold-all D by

$$Y_s \circ T_s = P(y(\Omega_s) \circ T_s) \tag{21}$$

As $[s \mapsto Y_s \circ T_s] \in C^0(I; H^{n+1}(D; \mathbb{R})) \cap C^1(I; H^n(D; \mathbb{R}))$, $[s \mapsto Y_s] \in C^1(I; H^n(D; \mathbb{R}))$ and

$$\frac{\partial Y_s}{\partial s}(0) = \frac{\partial Y_s \circ T_s}{\partial s}(0) - \nabla Y_0 \cdot V(0)$$

(direct adaptation of [9, prop 2.38, p.71]), y' is given as the restriction:

$$y' = \frac{\partial Y_s}{\partial s}(0) \Big|_{\Omega}$$

Notice that the preceding construction *a priori* requires that $y(\Omega)$ is uniquely defined when Ω is given. This construction easily extends to the case of multiple solutions $y(\Omega)$ when the choice of a branch $[s \mapsto y(\Omega_s)]$ has been made.

Theorem 3 *Under the assumptions* **A1** *and* **A2**, *the branch of solutions u of the Navier-Stokes problem considered in theorem 2 is shape differentiable in H^m at Ω_0 and satisfies property iii) with $n = m$. Moreover, the associated pressure p is shape differentiable in H^{m-1} at Ω_0 and satisfies iii) with $n = m - 1$.*

Proof: The desired regularity of $[s \mapsto u(\Omega_s \circ T_s)]$ is a direct consequence of the regularity of $[s \mapsto u^s]$, given in the theorem 2, and of the one of $[s \mapsto C_s]$, described in the remark 2. The transported pressure also has analogous regularity: its gradient being given by $\nabla p^s = \nabla(p_s \circ T_s) = -\nu A^s u^s - B^s(u^s, u^s) + f^s$, the mapping $[s \mapsto \nabla p^s] \in C^0(I; H^{m-1}(D; \mathbb{R}^p)) \cap C^1(I; H^{m-2}(D; \mathbb{R}^p))$. The fields p_s (and p^s) are defined up to a constant. We may set, for example, $\int_{\Omega_0} p_s \circ T_s \, dx = 0$, and get with that choice the regularity $[s \mapsto p_s \circ T_s] \in C^0(I; H^m(D; \mathbb{R}^p)) \cap C^1(I; H^{m-1}(D; \mathbb{R}^p))$. ■

4.2 Tangential operators

We refer to [9] for the definition of the following usual tangential operators on the boundary Γ of a C^2 open bounded set $\Omega \subset \mathbb{R}^3$:

- the tangential gradient $\nabla_\Gamma : H^1(\Gamma; \mathbb{R}) \to L^2(\Gamma; \mathbb{R}^3)$
- the tangential divergence $\mathrm{div}_\Gamma : H^1(\Gamma; \mathbb{R}^3) \to L^2(\Gamma; \mathbb{R})$
- the Laplace-Beltrami operator $\Delta_\Gamma : H^2(\Gamma; \mathbb{R}) \to L^2(\Gamma; \mathbb{R})$

These tangential operators are connected to the corresponding operators in Ω: for any $y \in C^1(\overline{\Omega}; \mathbb{R})$, $v \in C^1(\overline{\Omega}; \mathbb{R}^3)$ and $z \in C^2(\overline{\Omega}; \mathbb{R})$, we have

$$\nabla_\Gamma y = \nabla y|_\Gamma - \frac{\partial y}{\partial n} n \text{ and } \mathrm{div}_\Gamma v = \mathrm{div} v|_\Gamma - [\mathrm{D}v]n \cdot n \tag{22}$$

$$\Delta_\Gamma z = \Delta z|_\Gamma - \kappa \frac{\partial z}{\partial n} - \frac{\partial^2 z}{\partial n^2} \tag{23}$$

where κ is the mean curvature of Γ. By density, equations (22) still hold for $y \in H^{\frac{3}{2}+\varepsilon}(\Omega; \mathbb{R})$, $v \in H^{\frac{3}{2}+\varepsilon}(\Omega; \mathbb{R}^3)$ and equation (23) for $z \in H^{\frac{5}{2}+\varepsilon}(\Omega; \mathbb{R})$, $\varepsilon > 0$.

Many tangential calculus formulas are shown easily with the use of the equations (22) to (23) for smooth functions and then extended by density. In particular, we will need the identity

$$\mathrm{div}_\Gamma(yv) = y \, \mathrm{div}_\Gamma v + \nabla_\Gamma y \cdot v \tag{24}$$

which is true for any $y \in C^1(\overline{\Omega}; \mathbb{R})$ and $v \in H^{\frac{3}{2}+\varepsilon}(\Omega; \mathbb{R}^3)$.

Proposition 5 (Integration by part on Γ) *For all $(y, v) \in H^1(\Gamma; \mathbb{R}) \times H^1(\Gamma; \mathbb{R}^3)$ we have*

$$\int_\Gamma \nabla_\Gamma y \cdot v \, d\Gamma = -\int_\Gamma y \, \mathrm{div}_\Gamma v \, d\Gamma + \int_\Gamma \kappa y (v \cdot n) d\Gamma \tag{25}$$

Lemma 2 *Assume that $u \in H_0^1(\Omega; \mathbb{R}^3) \cap H^2(\Omega; \mathbb{R}^3)$. Then, we have*

$$\mathrm{D}u = [\mathrm{D}u] n n^* \text{ on } \Gamma \tag{26}$$

Moreover, $\mathrm{div}\, u = 0$ *in Ω (that is, $u \in V^2(\Omega)$) implies that*

$$[\mathrm{D}u] n \cdot n = 0 \text{ and } [\mathrm{D}u]^* n = 0 \text{ on } \Gamma \tag{27}$$

Proof: Let $y \in H_0^1(\Omega; \mathbb{R}) \cap H^2(\Omega; \mathbb{R})$. As $y|_\Gamma = 0$, $\nabla_\Gamma y = 0$ and the equation (22) leads to $\nabla y|_\Gamma = \frac{\partial y}{\partial n} n$. This result, used for $y = u^i$, $1 \leq i \leq 3$ proves the first part of the lemma. The second part is proved as follows: we use the decomposition (22) on the boundary. As $u|_\Gamma = 0$, $\mathrm{div}_\Gamma\, u = 0$ and we conclude that $[\mathrm{D}u] n \cdot n = 0$ on Γ. As $[\mathrm{D}u] = [\mathrm{D}u] n n^*$, $[\mathrm{D}u]^* = n n^* [\mathrm{D}u]^*$ and $[\mathrm{D}u]^* n = n(n^* [\mathrm{D}u]^* n) = n(n^* [\mathrm{D}u] n) = 0$. ■

4.3 The boundary shape derivative

We may now define the boundary shape derivative of a mapping $\Omega \mapsto y(\Omega)$ which is shape differentiable in H^m, $m \geq 1$ (resp. in C^m, $m \geq 0$): it is the element of $H^{m-\frac{1}{2}}(\Gamma; \mathbb{R})$ (resp. of $C^m(\Gamma; \mathbb{R})$), defined by

$$y'_\Gamma = \dot{y}|_\Gamma - \nabla_\Gamma y \cdot V(0) \tag{28}$$

Obviously, using the equations (20) and (22), we have also

$$y'_\Gamma = y'|_\Gamma + \frac{\partial y}{\partial n} (V(0) \cdot n) \tag{29}$$

As the material derivative and tangential gradient which appear in formula (28) only depends on the values of y on the boundary Γ of Ω, we may more generally define the boundary shape derivative of a mapping $\Gamma \mapsto y(\Gamma)$ such that there is an extension of y in $H^m(\Omega; \mathbb{R})$ (resp. $C^m(\overline{\Omega}; \mathbb{R})$) which is shape differentiable.

This is the way the boundary shape derivative of n may be defined (*via* a shape differentiable C^2 extension of n in Ω) and we have (see [2])

$$n'_\Gamma = -\nabla_\Gamma(V(0)\cdot n) \tag{30}$$

For two mappings y and z respectively shape differentiable in H^1 and in C^1, the product is shape differentiable in H^1 and its boundary shape derivative is given by

$$(yz)'_\Gamma = (y)'_\Gamma z + y(z)'_\Gamma \tag{31}$$

Proposition 6 *Assume that y is shape differentiable in H^1. Then*

$$\frac{\partial}{\partial s}\left(\int_{\Gamma_s} y(\Gamma_s)\, d\Gamma\right)\Big|_{s=0} = \int_\Gamma y'_\Gamma\, d\Gamma + \int_\Gamma \kappa y(V(0)\cdot n)\, d\Gamma \tag{32}$$

4.4 PDE satisfied by u'. Adjoint Equation.

The assumption **A2** implies that the linearized Navier-Stokes problem with a right-hand side in $H^{-1}(\Omega_0;\mathbb{R}^3)$ and homogeneous Dirichlet boundary condition has a unique solution $u \in V^1(\Omega_0)$. The operator $L : V^1(\Omega_0) \to W^{-1}(\Omega_0)$ involved in abstract formulation $L(u) = \pi(g)$ of this equation being an isomorphism, the adjoint operator $L^\star : V^1(\Omega_0) \to W^{-1}(\Omega_0)$ is also an isomorphism (we made the identification $W^{-1}(\Omega_0) \simeq V^1(\Omega_0)'$). The PDE form of the adjoint equation $L^\star(\eta) = \pi(h)$ is $-\nu\Delta\eta + [\mathrm{D}u]^*\eta - [\mathrm{D}\eta]u + \nabla\zeta = h$ for $\eta \in V^1(\Omega_0)$ and $h \in H^{-1}(\Omega_0;\mathbb{R}^3)$. The well-posedness of the two problems

$$1)\begin{cases} -\nu\Delta v + [\mathrm{D}u]v + [\mathrm{D}v]u + \nabla q = 0 \\ \operatorname{div} v = 0 \\ v|_\Gamma = h \end{cases} \quad 2)\begin{cases} -\nu\Delta\eta + [\mathrm{D}u]^*\eta - [\mathrm{D}\eta]u + \nabla\zeta = 0 \\ \operatorname{div}\eta = 0 \\ \eta|_\Gamma = h \end{cases} \tag{33}$$

for $g \in H^{-1}(\Omega_0;\mathbb{R}^3)$, $h \in H^{1/2}(\Gamma;\mathbb{R}^3)$ follows directly from the existence of $g \in V^1(\Omega_0)$ such that $g|_\Gamma = h$ if h satisfies the compatibility condition

$$\int_\Gamma h\cdot n\, d\Gamma = 0 \tag{34}$$

(remember that Ω_0 is connected). Moreover, for both equations, and from the regularity of u implied by **A2**, the solutions v and η corresponding to a boundary condition $h \in H^{3/2}(\Gamma;\mathbb{R})$ are in $H^2(\Omega_0;\mathbb{R})$.

The problems (33.1) and (33.2) are used intensively in the calculations of shape derivatives because of the following result:

Proposition 7 *The couple (u', p') is the only solution (v, q) of the linearized Navier-Stokes equation at u (33.1) with the boundary condition $v|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$.*

Proof: First we notice that $[\mathrm{D}u]n(V(0)\cdot n)\cdot n$ being zero on the boundary (see (27)), the compatibility condition (34) is satisfied. Moreover, for any $\varphi \in \mathcal{D}(\Omega_0;\mathbb{R}^3)$, for any $|s|$ being small enough, $\varphi \in \mathcal{D}(\Omega_s;\mathbb{R}^3)$ and any solution u_s of the Navier-Stokes equation in Ω_s satisfies

$$\int_{\Omega_s}(-\nu\Delta u_s + [\mathrm{D}u_s]u_s + \nabla p_s - f)\cdot\varphi\,dx = \int_D(-\nu\Delta U_s + [\mathrm{D}U_s]U_s + \nabla P_s - f)\cdot\varphi\,dx \;=\; 0$$

where $[s\mapsto U_s] \in C^1(I; H^2(D;\mathbb{R}^3))$ and $[s\mapsto P_s]\in C^1(I;H^1(D;\mathbb{R}))$ are the extensions considered in the equation (21). The differentiation of this equation with respect to s gives

$$\int_D(-\nu\Delta U' + [\mathrm{D}U']U + [\mathrm{D}U]U' + \nabla P')\cdot\varphi\,dx = 0$$

which implies that u' and p' are solutions of (33.1) . The boundary condition is proved as follows: as $u|_\Gamma = 0$, for any $\varphi\in C^\infty(\overline{D};\mathbb{R}^3)$,

$$\int_{\Gamma_s} u\cdot\varphi\,d\Gamma = 0$$

using equation (32), we differentiate this equation for $s=0$. As $(u\cdot\varphi)'_\Gamma = u'_\Gamma\cdot\varphi$ and $u'_\Gamma = u'|_\Gamma + [\mathrm{D}u]n(V(0)\cdot n)$, we obtain

$$\int_\Gamma (u' + [\mathrm{D}u]n(V(0)\cdot n))\cdot\varphi\,d\Gamma = 0$$

which implies that $u'|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$. ■

The calculation of the shape derivative of the functional considered in the next section also requires the following equations, given for divergence-free fields v and η in $H^2(\Omega;\mathbb{R}^3)$ and $u\in V^1(\Omega)$ by the repeated use of Green's formula.

$$\int_\Omega -\nu\Delta\eta\cdot v\,dx = \int_\Omega \eta\cdot(-\nu\Delta v)\,dx + \nu\int_\Gamma([\mathrm{D}v]n\cdot\eta - [\mathrm{D}\eta]n\cdot v)\,d\Gamma \tag{35}$$

$$\int_\Gamma [\mathrm{D}u]^*\eta\cdot v\,dx = \int_\Omega \eta\cdot[\mathrm{D}u]v\,dx \text{ and } \int_\Omega -[\mathrm{D}\eta]u\cdot v = \int_\Omega \eta\cdot[\mathrm{D}v]u\,dx \tag{36}$$

$$\int_\Omega \nabla\zeta\cdot v\,dx = \int_\Gamma \zeta(v\cdot n)\,d\Gamma \text{ and } \int_\Omega \eta\cdot\nabla q\,dx = \int_\Gamma(\eta\cdot n)q\,d\Gamma \tag{37}$$

5 Shape derivative of sharp functionals

Geometrical Setting: In this section, the boundary Γ of Ω is the union of two disjoint 2−dimensional submanifolds Γ_α and Γ_β. Let F and M be respectively the resultant of the forces and the moment of the forces exerted by the fluid on a piece Γ_α. The vectors F and M are given by

$$F = \int_{\Gamma_\alpha} (-\sigma(u)n + pn)\, d\Gamma \quad \text{and} \quad M = \int_{\Gamma_\alpha} x \times (-\sigma(u)n + pn)\, d\Gamma \tag{38}$$

with $\sigma(u) = \nu([\mathrm{D}u]^* + [\mathrm{D}u])$. Our aim in this section is to prove that $F(\Omega)$ and $M(\Omega)$ are *shape differentiable functionals* and to find the expression of their *Eulerian derivative.* We recall (see [9]) that a functional $J(\Omega)$ is shape differentiable at Ω_0 if for any V regular enough, $s \mapsto J(\Omega_s)$ is differentiable at $s = 0$ and this derivative (the Eulerian derivative) $dJ(\Omega_0; V)$ is linear continuous in V.

Theorem 4 *Under the hypotheses* **A1** *with m=2 and* **A2***, both F and M are shape differentiable at Ω_0. For a given $e \in \mathbb{R}^3$, let η be the solution of the adjoint equation (33.2) with the boundary conditions $\eta|_{\Gamma_\alpha} = e$ and $\eta|_{\Gamma_\beta} = 0$. (resp. $\eta|_{\Gamma_\alpha} = e \times x$ and $\eta|_{\Gamma_\beta} = 0$).*

Then the Eulerian derivative of $F_e = F \cdot e$ (resp. $M_e = M \cdot e$) is given by:

$$dF_e(\Omega_0; V) = \int_{\Gamma_\alpha} (f \cdot e)(V(0) \cdot n)\, d\Gamma + \nu \int_\Gamma [\mathrm{D}\eta]n \cdot [\mathrm{D}u]n(V(0) \cdot n)\, d\Gamma \tag{39}$$

$$\text{resp.} \quad \begin{aligned} dM_e(\Omega_0; V) = &\int_{\Gamma_\alpha} (x \times f + \nu[\mathrm{D}u]n \times n) \cdot e\, (V(0) \cdot n)\, d\Gamma \\ &+\nu \int_\Gamma [\mathrm{D}\eta]n \cdot [\mathrm{D}u]n(V(0) \cdot n)\, d\Gamma \end{aligned} \tag{40}$$

Let e be a given vector of $\mathbb{R}^3$ and $r \in C^\infty(D; \mathbb{R})$ a function such that $r = 1$ in a neighbourghood of Γ_α and $r = 0$ in a neighbourghood of $\Gamma_\beta = \Gamma - \Gamma_\alpha$. We set $g_F = re$ and $g_M = re \times x$. The values F_e and M_e may be studied as special cases of the family of functionals $J(\Omega) = J_1(\Omega) - J_2(\Omega)$ where

$$J_1(\Omega) = \int_\Gamma pn \cdot g\, d\Gamma, \;\; J_2(u) = \int_\Gamma \sigma(u)n \cdot g\, d\Gamma$$

and g satisfies $\operatorname{div} g = 0$ in a neighbourghood of Γ and the compatibility condition (34). It is obvious that with $g = g_F$, $J(\Omega) = F_e$. Moreover, as $(x \times (-\sigma(u)n + pn)) \cdot re =$

$(-\sigma(u)n+pn)\cdot(re\times x)$, $J(\Omega)=M_e$ with $g=g_M$. The two additional properties come easily from the equations $\operatorname{div} e=0$ and $\operatorname{div}(e\times x)=\operatorname{rot}(e)\cdot x-\operatorname{rot}(x)\cdot e=0$.

Shape Derivative of $J_1(\Omega)$: using the shape differentiability of p, g and n, and the proposition 6, we obtain

$$dJ_1(\Omega_0;V) \;=\; \int_\Gamma p'_\Gamma(g\cdot n)\,d\Gamma+\int_\Gamma p(g\cdot n)'_\Gamma\,d\Gamma+\int_\Gamma \kappa(pn\cdot g)(V(0)\cdot n)\,d\Gamma$$

As $p'_\Gamma=p'|_\Gamma+\frac{\partial p}{\partial n}(V(0)\cdot n)$ (see equation (29)), we have $dJ_1(\Omega;V)=A+\int_\Gamma p'(g\cdot n)\,d\Gamma$ with

$$A=\int_\Gamma\left(\frac{\partial p}{\partial n}(g\cdot n)(V(0)\cdot n)+p(g\cdot n)'_\Gamma+\kappa(pn\cdot g)(V(0)\cdot n)\right)\,d\Gamma$$

- As $g'_\Gamma=g'|_\Gamma+[\mathrm{D}g]n(V(0)\cdot n)$ (straightforward adaptation of (29) to the vectorial case) and $n'_\Gamma=-\nabla_\Gamma(V(0)\cdot n)$ (equation (30)), we have $(g\cdot n)'_\Gamma=g'_\Gamma\cdot n+g\cdot n'_\Gamma=([\mathrm{D}g]n\cdot n)(V(0)\cdot n)-g\cdot\nabla_\Gamma(V(0)\cdot n)$. Using the integration by parts formula on the boundary (25), we get

$$-\int_\Gamma pg\cdot\nabla_\Gamma(V(0)\cdot n)\,d\Gamma=\int_\Gamma \operatorname{div}_\Gamma(pg)(V(0)\cdot n)\,d\Gamma-\int_\Gamma\kappa(pn\cdot g)(V(0)\cdot n)\,d\Gamma$$

and finally, since $\operatorname{div} g=0$ on Γ,

$$\begin{aligned}A &= \int_\Gamma\left(\frac{\partial p}{\partial n}(g\cdot n)+p[\mathrm{D}g]n\cdot n+\operatorname{div}_\Gamma(pg)\right)(V(0)\cdot n)\,d\Gamma\\ &= \int_\Gamma \operatorname{div}(pg)(V(0)\cdot n)\,d\Gamma=\int_\Gamma\nabla p\cdot g(V(0)\cdot n)\,d\Gamma\end{aligned}$$

- We set $h=(g\cdot n)n$. The compatibility condition (34) being satisfied, we may introduce η_1, the unique solution of the adjoint equation (33.2). We have then

$$\begin{aligned}\int_\Gamma p'n\cdot g\,d\Gamma &= \int_\Gamma(p'\eta_1)\cdot n\,d\Gamma\\ &= \int_\Omega\operatorname{div}(p'\eta_1)\,dx\\ &= \int_\Omega\nabla p'\cdot\eta_1\,dx\quad(\text{as }\operatorname{div}\eta_1=0)\\ &= \int_\Omega(\nu\Delta u'-[\mathrm{D}u]u'-[\mathrm{D}u']u)\cdot\eta_1\,dx\end{aligned}$$

Using equations (35) and (36) we come to

$$\int_\Gamma p'n\cdot g\,d\Gamma=\int_\Omega(\nu\Delta\eta_1-[\mathrm{D}u]^*\eta_1+[\mathrm{D}\eta_1]u)\cdot u'\,dx+\nu\int_\Gamma([\mathrm{D}u']n\cdot\eta_1-[\mathrm{D}\eta_1]n\cdot u')\,d\Gamma$$

As $\nu\Delta\eta_1 - [\mathrm{D}u]^*\eta_1 + [\mathrm{D}\eta_1]u = \nabla\zeta_1$, equation (37) yields

$$\int_\Omega (\nu\Delta\eta_1 - [\mathrm{D}u]^*\eta_1 + [\mathrm{D}\eta_1]u)\cdot u'\,dx = \int_\Omega \nabla\zeta_1\cdot u'\,dx = \int_\Gamma \zeta_1(u'\cdot n)\,d\Gamma = 0$$

due to $u'|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$ and $[\mathrm{D}u]n\cdot n = 0$ (see Lemma 2). On the other hand, $[\mathrm{D}u']n\cdot\eta_1 = (g\cdot n)[\mathrm{D}u']n\cdot n$ and on the boundary, $0 = \operatorname{div} u' = \operatorname{div}_\Gamma u' + [\mathrm{D}u']n\cdot n$. Accordingly, we have by integration by parts on Γ and because $u' = u'_\tau$

$$\nu\int_\Gamma [\mathrm{D}u']n\cdot\eta_1\,d\Gamma = -\nu\int_\Gamma (g\cdot n)\operatorname{div}_\Gamma u'\,d\Gamma = \nu\int_\Gamma \nabla_\Gamma(g\cdot n)\cdot u'\,d\Gamma \tag{41}$$

- Finally, since $u'|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$, we end up with

$$dJ_1(\Omega_0;V) = \int_\Gamma (\nabla p\cdot g + \nu([\mathrm{D}\eta_1]n - \nabla_\Gamma(g\cdot n))\cdot[\mathrm{D}u]n)\,(V(0)\cdot n)\,d\Gamma \tag{42}$$

Shape Derivative of $J_2(\Omega)$: we notice that lemma 2 implies that $\sigma(u)n = \nu[\mathrm{D}u]n$. The shape differentiability of n, g and u (in H^2 for the latter) implies that

$$dJ_2(\Omega_0;V) = \int_\Gamma \nu([\mathrm{D}u]n\cdot g)'_\Gamma\,d\Gamma + \int_\Gamma \kappa\nu([\mathrm{D}u]n\cdot g)(V(0)\cdot n)\,d\Gamma$$

As $([\mathrm{D}u]n\cdot g)'_\Gamma = [\mathrm{D}u]'_\Gamma n\cdot g + [\mathrm{D}u]n'_\Gamma\cdot g + [\mathrm{D}u]n\cdot g'_\Gamma$ and $[\mathrm{D}u]'_\Gamma = [\mathrm{D}u']|_\Gamma + [\mathrm{D}^2u]n(V(0)\cdot n)$, we have $dJ_2(\Omega;V) = B + \int_\Gamma \nu[\mathrm{D}u']n\cdot g_\tau\,d\Gamma$ with $g_\tau = g - (g\cdot n)n$ and

$$B = \int_\Gamma \nu\left((g\cdot n)[\mathrm{D}u']n\cdot n + (n^*[\mathrm{D}^2u]n\cdot g + [\mathrm{D}g]n\cdot[\mathrm{D}u]n + \kappa[\mathrm{D}u]n\cdot g)(V(0)\cdot n)\right)\,d\Gamma$$

- The first term of B has already been calculated (see equation (41)) . The second is involved in the decomposition of the Laplace operator on the boundary (23). As $\Delta_\Gamma u = 0$ and $[\mathrm{D}u]u{=}0$ on Γ, $\nu n^*[\mathrm{D}^2u]n = -\nu\kappa[\mathrm{D}u]n + \nabla p - f$ on Γ. Therefore,

$$B = \int_\Gamma ((\nabla p - f)\cdot g + \nu[\mathrm{D}g]n\cdot[\mathrm{D}u]n - \nu\nabla_\Gamma(g\cdot n))\,(V(0)\cdot n)\,d\Gamma$$

- We set $h = g_\tau$. The compatibility condition (34) being satisfied, we may introduce η_2, the unique solution of the adjoint equation (33.2). The equation (35) yields

$$\nu\int_\Gamma [\mathrm{D}u']n\cdot g_\tau\,d\Gamma = \int_\Omega (-\nu\Delta\eta_2\cdot u' + \eta_2\cdot(\nu\Delta u'))\,dx + \nu\int_\Gamma [\mathrm{D}\eta_2]n\cdot u'\,d\Gamma$$

Using $-\nu\Delta\eta_2 = -[\mathrm{D}u]^*\eta_2 + [\mathrm{D}\eta_2]u - \nabla\zeta$ and $\nu\Delta u' = [\mathrm{D}u]u' + [\mathrm{D}u']u + \nabla p'$, and then the equations (36) to (37), we come to

$$\nu\int_\Gamma [\mathrm{D}u']n\cdot g_\tau\,d\Gamma \;=\; \int_\Gamma \zeta(u'\cdot n)\,d\Gamma + \int_\Gamma (\eta_2\cdot n)p'\,d\Gamma + \nu\int_\Gamma [\mathrm{D}\eta_2]n\cdot u'\,d\Gamma$$

and the two first terms of the right-hand side vanish.

- We finally have

$$dJ_2(\Omega_0; V) = \int_\Gamma \left((\nabla p - f) \cdot g + \nu[\mathrm{D}g]n \cdot [\mathrm{D}u]n - \nu\nabla_\Gamma(g \cdot n)\right)(V(0) \cdot n)\, d\Gamma \qquad (43)$$

Shape Derivative of $J(\Omega)$**:** with the expressions (42) and (43), we come to

$$dJ(\Omega_0; V) = \int_\Gamma (f \cdot g)(V(0) \cdot n) + \nu \int_\Gamma [\mathrm{D}(\eta - g)]n \cdot [\mathrm{D}u]n(V(0) \cdot n)\, d\Gamma \qquad (44)$$

where $\eta = \eta_1 + \eta_2$ is the unique solution of the equation (33.2) with the boundary condition $\eta|_\Gamma = g$. This expression easily leads to the formulas (39) and (40) of the theorem 4 with the choices of $g = g_F$ and $g = g_M$.

6 Conclusion

The natural extension of this work is the study of the shape derivative of functionals that require even more regularity than the one considered in the section 5. For example for the functionals that characterize the uniformity of the pressure on a given body, the shape differentiability of p in H^1 (which corresponds to $m = 2$) is not sufficient. Notice also that with a slight adaptation, the case of non-homogeneous boundary conditions, which appear for example in the study of a body which moves in a fluid with a constant velocity, also fits in the preceding framework.

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